Eur. Phys. J. B **46**, 449–455 (2005)
DOI: 10.1140/epjb/e2005-00273-5

THE EUROPEAN PHYSICAL JOURNAL B

An interpretation of the She-Lévêque model based on order statistics

P. St-Jean[a]

Laboratoire Joliot-Curie, École Normale Supérieure de Lyon, 46 Allée d'Italie, 69364 Lyon Cedex 07, France

Received 25 February 2005
Published online 18 August 2005 –

Abstract. We present an interpretation of the She-Lévêque model in fully developed turbulence based on order statistics. Turbulent behavior at large values of the Reynolds number is often studied through the scaling behavior of moments of the distribution of the velocity differences and of the energy dissipation. The present interpretation leads to a derivation of the scaling exponents ζ_p and τ_p of these moments, without any postulate about a universal relation over the fluctuation structures such as the one used by She and Lévêque. The interpretation is based on the fact that the hierarchy of fluctuation structures imposes statistical constraints, whereupon the order p itself is seen as a random variable. As proposed by She and Lévêque, the hierarchy of the structures is such that the structures of larger order affect locally lower order structures through an entrainment process. This phenomenon leads to the Fisher-Tippett law, one of three asymptotic distributions for the extreme value of a random sample as the size of the sample grows to infinity.

PACS. 47.27.Jv High-Reynolds-number turbulence – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

For the last sixty years or so, the phenomenological study of fully developed turbulence has evolved from the celebrated work of Kolmogorov [1] to the recent general acceptance that significant discrepancy from his early work exists, the effect known as intermittency. This effect only appears when rare outlier events occur, forcing the need for gigantic statistics in order to be established properly. The study of scaling exponents (of high-order moments) of the velocity increments constitutes the obvious path in quantifying the presumed discrepancies. The scaling behavior of the longitudinal velocity increments, $\delta v_l = v(x+l) - v(x)$, is expressed as

$$\langle |\delta v_l|^p \rangle \sim l^{\zeta(p)}. \tag{1}$$

The Kolmogorov refined similarity hypothesis [2] links the behavior of $\zeta(p)$ to the scaling behavior of the energy dissipation over a ball of size l, ε_l, through the following relation:

$$\langle |\delta v_l|^p \rangle \sim \langle \varepsilon_l^{p/3} \rangle l^{p/3}. \tag{2}$$

Defining $\tau(p)$ as the scaling exponent of ε_l, the relation becomes

$$\zeta(p) = p/3 + \tau(p/3). \tag{3}$$

[a] e-mail: `philippe.saint-jean@ens-lyon.fr`

The original work of Kolmogorov [1] (later referred to as K41) led to the prediction that ε_l was independent of l, with the obvious consequences that $\tau(p) = 0$ and $\zeta(p) = p/3$. The observed discrepancies from this behavior of the scaling exponents are referred to as *intermittency* [3,4]. The term indicates the fact that non-linear corrections to $\zeta(p)$ originate from intermittent behavior of the energy dissipation.

Over the past thirty years a number of phenomenological models have been proposed in order to account for non-linear corrections to $\zeta(p)$ [5,6]. Perhaps the most "popular" model was proposed ten years ago by She and Lévêque [7]. It won notoriety not only because it best fits experimental data for large values of p [8], but mostly because it relies on an interesting physical picture that leads in a somehow natural fashion to an explicit form for the scaling exponents, namely $\tau(p) = -2p/3 + 2[1 - (2/3)^p]$. Dubrulle [9] put this model in the context of log-infinitely divisible cascade models [9–12] and realized that this form of τ_p corresponds to a log-Poisson distribution along the steps of the cascade. This was presented independently by She and Waymire [12]. However the She-Lévêque model relies on a *ad hoc* postulate of this explicit form. This postulate can be interpreted as the symptom of a missing link between a valid physical model (which includes constraints that have not been recognized yet) and a large set of experimental results.

The intent of this study is to provide this missing link through a statistical argument from which the explicit form for the scaling exponents can be derived rather than postulated.

In Section 2 we give a brief presentation of the She-Lévêque model. Section 3 investigates the underlying statistical constraint imposed by the She-Lévêque model that leads to the explicit form for $\tau(p)$ and $\zeta(p)$. In Section 4 a more general result is obtained from fewer hypothesis than in the She-Lévêque model.

2 The She-Lévêque model

The model proposed by She and Lévêque [7] characterizes the energy dissipation field ε_l using a hierarchy of fluctuation structures $\varepsilon_l^{(p)}$ defined by the ratio of successive moments of ε_l, i.e.

$$\varepsilon_l^{(p)} = \frac{\langle \varepsilon_l^{p+1} \rangle}{\langle \varepsilon_l^p \rangle}. \tag{4}$$

The two extreme structures $\varepsilon_l^{(0)}$ and $\varepsilon_l^{(\infty)}$ correspond respectively to the mean fluctuation structure $\bar{\varepsilon}$ and to filaments [13,14], in view of the two-fluid model [15]. Intermediate values of p thus represent more or less organized fluid structures, perhaps ribbon or sheetlike.

It is argued in the She-Lévêque model that the filamentary structures $\varepsilon_l^{(\infty)}$ scale like $l^{-2/3}$. From the definition of $\tau(p)$, this amounts in stating that as $p \to \infty$,

$$\tau(p+1) - \tau(p) \to -2/3, \tag{5}$$

hence $\tau(p) \to -2p/3 + C$. Using Legendre transform the constant C is readily seen as the codimension [5] of the high intensity structures (filaments), hence $C = 3 - 1 = 2$.

Perhaps the most important part of the model lies in the assumed interaction between structures of different order. She-Lévêque propose that the fluctuation structures $\varepsilon_l^{(p)}$ form a hierarchy in the sense that high order structures entrain lower order ones nearby. This phenomenon is explained in greater detail in the next section.

From the supposed hierarchy of those structures, they postulate the following relation between structures of adjacent order:

$$\varepsilon_l^{(p+1)} = A_p {\varepsilon_l^{(p)}}^{\beta} {\varepsilon_l^{(\infty)}}^{1-\beta}. \tag{6}$$

From equations (4, 6), we get the following expression for τ_p, the scaling exponent of $\langle \varepsilon_l^p \rangle$:

$$\tau_{p+2} - (1+\beta)\tau_{p+1} + \beta\tau_p + \frac{2}{3}(1-\beta) = 0. \tag{7}$$

The most intermittent structures $\varepsilon_l^{(\infty)}$ scales like $l^{-2/3}$, which in returns implies that τ_p must be of the form $\tau_p = -2/3p + 2 + f(p)$, where $f(\infty) = 0$. We can then rewrite equation (7) as

$$f(p+2) - (1+\beta)f(p+1) + \beta f(p) = 0. \tag{8}$$

This last expression has $f(p) = \alpha\beta^p$ as its only nontrivial solution, and boundary conditions on τ_p (namely $\tau_0 = \tau_1 = 0$) leads to

$$\tau_p = -\frac{2}{3}p + 2\left[1 - \left(\frac{2}{3}\right)^p\right]. \tag{9}$$

Although the derivation of the form of τ_p starting with equation (6) is straightforward, the latter was postulated in a somewhat *ad hoc* fashion. Bear in mind that this form of τ_p corresponds to a log-Poisson cascade [9,12]. However the form of τ_p was obtained by She and Lévêque without requiring the *cascade through scales* picture.

In the next section we propose a statistical interpretation of the She-Lévêque model which shows that the underlying physics of the model forces the universal scaling behavior without having to formulate a postulate regarding its explicit form, given by equation (6). Still no explicit formulation of the cascade picture is required, staying in the same line as the She-Lévêque model.

3 The order parameter as a hidden variable

In this section we propose an interpretation of the hierarchy of fluctuation structures that we believe was implicitly present in the original work of She and Lévêque. It appears that the order p of a structure can be seen as an index of different dynamical modes of dissipation. High values of p correspond to more coherent modes. This notion is explicitly stated in [7]. We propose that p should be interpreted as a random variable itself. Our plan is then to interpret the She-Lévêque model in order to obtain constraints on the distribution of p. This idea is useful provided that one can relate p to the dissipation through a conditional probability, in such a way that the form of $\tau(p)$ can be obtained.

We focus on an additional comment made by She and Lévêque in [7]. They indicate that the intensities of the fluctuation structures can be rewritten as:

$$\varepsilon_l^{(p)} = \frac{\int \varepsilon_l^{p+1} P(\varepsilon_l) d\varepsilon_l}{\int \varepsilon_l^p P(\varepsilon_l) d\varepsilon_l} = \int \varepsilon_l Q_p(\varepsilon_l) d\varepsilon_l, \tag{10}$$

where $P(\varepsilon_l)$ is the probability density function (pdf) of ε_l and

$$Q_p(\varepsilon_l) = \frac{P_{\varepsilon_l}(\varepsilon_l) \cdot \varepsilon_l^p}{\langle \varepsilon_l^p \rangle}. \tag{11}$$

Obviously $Q_p(\varepsilon_l)$ are also valid pdfs of ε_l, for which the intensities $\varepsilon_l^{(p)}$ are the mathematical expectations. Let us look at Q_p in more details. If we inspect the behavior of Q_p for dissipation values which scale with exponent $h(p)$, i.e. for $\varepsilon_l = \varepsilon_0 \cdot l^{h(p)}$, where $h(p)$ is the singularity exponent associated to p in the multi-fractal formalism, one gets

$$Q_p(\varepsilon_0 \cdot l^{h(p)}) \sim P_{\varepsilon_l}(\varepsilon_0 \cdot l^{h(p)}) \cdot l^{p \cdot h(p) - \tau(p)}. \tag{12}$$

Since the fractal dimension $F(h(p))$ for the support of singularities of exponent $h(p)$ is given by

$F(h(p)) = p \cdot h(p) - \tau(p)$, we can view Q_p as the nondimensionalized pdf (in a fractal sense) of the dissipation for a given scaling exponent $h(p)$. This interpretation was not formulated in those terms in [7]. Note that this last expression is only valid in the limit $l \to 0$.

From this point of view, it then seems natural to look at this family of pdfs as conditional distributions of ε_l over the value of a hidden parameter p. We believe this is a more formal understanding of the She-Lévêque hierarchy of fluctuation structures and that this notion was implicitly present in their work.

However, it is not possible to simply interpret $Q_p(\varepsilon_l)$ as the conditional distribution of the dissipation at a fixed value $p = p_0$. Indeed, by Bayes' theorem:

$$P_{\varepsilon_l|p=p_0}(\varepsilon_l) = \frac{P_{\varepsilon_l}(\varepsilon_l) \cdot P_{p|\varepsilon_l}(p_0)}{P_p(p_0)}. \tag{13}$$

Assuming $Q_{p_0}(\varepsilon_l) = P_{\varepsilon_l|p=p_0}(\varepsilon_l)$ would imply that

$$P_{p|\varepsilon_l}(p) = \left(\frac{\varepsilon_l^p}{\langle\varepsilon_l^p\rangle}\right) \cdot P_p(p). \tag{14}$$

This last expression cannot be normalized independently of ε_l, regardless of the form of $P_p(p)$. Hence $P_{p|\varepsilon_l}(p)$ is not a valid pdf in this case.

However, we note that for $p = 0$ we have $Q_0(\epsilon_l) = P_{\varepsilon_l}(\varepsilon_l)$, i.e. the marginal distribution of ε_l itself. This seems to indicate that the condition would rather be expressed as $p \geq p_0$ instead of $p = p_0$, such that the case $p_0 = 0$ involves no condition at all and falls back on the marginal distribution $P_{\varepsilon_l}(\varepsilon_l)$. Coming back to the conditional distribution picture, we translate this remark in probabilistic term by assuming that $Q_{p_0}(\varepsilon_l) = P_{\varepsilon_l|p\geq p_0}(\varepsilon_l)$.

Reformulating Bayes' theorem, the conditional probability respects the following relation:

$$P_{\varepsilon_l|p\geq p_0}(\varepsilon_l) = \frac{P_{\varepsilon_l}(\varepsilon_l) \cdot (1 - \bar{P}_{p|\varepsilon_l}(p_0))}{1 - \bar{P}_p(p_0)}, \tag{15}$$

where $\bar{P}(\cdot)$ is the cumulative density function (cdf) of the subscript variable.

It is tempting to reassemble equation (11) and equation (15) such that $1 - \bar{P}_{p|\varepsilon_l}(p) = \varepsilon_l^p$. However this last expression does not qualify as a cdf as it is not properly normalized. In fact it can even grow to infinity as $p \to \infty$. Note that we can multiply both the numerator and the denominator in equation (11) by any constant, or even any function of p without altering the form of Q_p. The She-Lévêque model also assumes that (for finite l) the distribution of ε_l is bounded above by some value $\varepsilon_l^{(\infty)}$. Writing $1 - \bar{P}_{p|\varepsilon_l}(p) = (\varepsilon_l/\varepsilon_l^{(\infty)})^p$ ensures that the cdf properly goes to one as $p \to \infty$, without affecting $Q_p(\varepsilon_l)$. Correspondingly, we will have that

$$1 - \bar{P}_p(p) = \frac{\langle\varepsilon_l^p\rangle}{(\varepsilon_l^{(\infty)})^p} \tag{16}$$

for the marginal distribution of p. Equation (16) is all that we will need in order to derive an explicit expression for $\tau(p)$; we consider equation (16) as the starting point for the derivation presented in the next sections.

Before we move on, we add a comment concerning the last expression. We might expect the cdf of p to scale as $-F(h(p))$, assuming a one-to-one correspondence of p with $h(p)$. But this is not the case, as pointed out in [7]: the expected value of Q_p, $\varepsilon_l^{(p)}$, rather scales as

$$\tau_{p+1} - \tau_p = \int_p^{p+1} \tau'(p) \tag{17}$$

$$= \int_p^{p+1} h(p) \tag{18}$$

$$\sim \bar{h}(p). \tag{19}$$

The probability $P_p(p)$ scales as

$$\tau(p) - C_\infty \cdot p = -F(h(p)) + (h(p) - C_\infty) \cdot p. \tag{20}$$

Note however that for $p \to \infty$, $\bar{h}(p) \to h(p)$ and $P_p(p)$ also behaves accordingly as it tends towards $-F(h(p))$. This is in agreement with the picture of poorly organized structures of low order, where $\bar{h}(p)$ represents an average scaling behavior. As p grows, one can show that the variance of $Q_p(\varepsilon_l)$ tends to zero such that the identification $h(p)$ becomes asymptotically valid.

4 Statistical behavior of the hierarchical dynamics

One fundamental aspect of the She-Lévêque model lies in their proposed hierarchical dynamics of the fluctuation structures $\varepsilon_l^{(p)}$. She and Lévêque observe that the hierarchy of structures is the result of an entrainment process whereas high-order structures "constantly entrain surrounding, less ordered fluid elements". Hence the dynamics that they propose is one of the *survival of the fittest* over the fluctuation structures $\varepsilon_l^{(p)}$.

In Section 3 we have related the distribution of the order p to the moments $\langle\varepsilon_l^p\rangle$ through an hypothesis on the conditional distribution of the dissipation over the order p. We have implicitly assumed that the order p, just like the dissipation, is a random field over space, time and scale, such that we should write equation (16) more formally as:

$$\Pr(p_l(x,t) \geq p_0) = \frac{\langle\varepsilon_l^{p_0}\rangle}{(\varepsilon_l^{(\infty)})^{p_0}}. \tag{21}$$

We now come back to the *survival of the fittest* dynamics. The She-Lévêque model includes an explicit statement on the behavior of this entrainment process: since larger order structures quickly absorb lower order ones nearby, it is very unlikely that any structure of a given order p can be found spatially close to a structure of order much larger than p. Similarly in time it is very unlikely that a small order p manifests itself when there was a large order structure around an instant before.

Hence according to this picture the probability of having a p-order structure at a given point (x,t) in space and time is proportional to the probability of the maximum structure order found in a spatial and timewise neighborhood (of (x,t)) being close to p. This imposes strong constraints on the possible statistical distribution of the structures.

Precisely it means that the following can be written about the distribution:

$$\Pr(p_l(x,t) = p_0) \sim \Pr(\max_{(x',t') \in B_l(x,t)} p_l(x',t') = p_0) \quad (22)$$

where $\{(x',t') \in B_l(x,t)\}$ is some neighborhood around (x,t). Looking at $p_l(x,t)$ as a random field, the constraint then tells us that the pointwise probability distribution function (pdf) of $p_l(x,t)$ must be invariant when replaced by the pdf of its maximum value in a neighborhood of any point (x,t). Again it means that some time before there could not have been structures of order much higher than p around. However it allows for the entrainment process. This is in perfect agreement with the She-Lévêque model. Note that random injection of energy alters this process; still if a steady-state is reached, then it should obey the constraint described above.

4.1 Long-range correlations

At this point, we need to elaborate on correlation issues. The argument presented above implicitly assumes that there is no long range correlation between the structures. This seems wrong since we expect that the typical long range correlations of the velocity and dissipation fields should manifest themselves.

A well-known trick to manage long-range correlations is to regroup the statistics in appropriately chosen "quantiles" which are strongly correlated, such that the quantiles amongst themselves are no longer long-range correlated, see for instance [16]. Conveniently enough, this is just what the parameter p does: it separates the dissipation distribution in structures with well-defined expected values $\varepsilon_l^{(p)}$ for large values of p. For small values of p, bear in mind that Q_p represents all structures with $p' > p$ (such that the $\varepsilon_l^{(p)}$ are not expected to represent a given value of p); note also that small values of p correspond to poorly organized structures, i.e. with behavior similar to white noise, thus uncorrelated. In our mind, this last comment expresses the reason that lies beneath the pertinence of the hierarchy of structures presented by She and Lévêque.

Still this does not mean that the structures are completely uncorrelated. Indeed they must be according to the assumed entrainment process, yet this interaction (although "non-linear") remains short-ranged.

Let us now see what actual form of the pdf of p can be expected.

4.2 Max-stable laws

It is a well-known result in order statistics that there are only three forms of asymptotic distributions that respect equation (22) when the number of elements amongst which the maximum is taken is arbitrarily large. Indeed this is just the case here: not because it can be assumed that the neighborhood is of a very large extent spatially (which would require that a very large number of structures can affect a given structure at a fixed time), but simply because the effect of taking maxima values as time goes by leads to the asymptotic distribution, i.e. the steady-state. Formally, we have the following result about these distributions:

Proposition 1 *The cumulative distribution function (cdf) of the maximum value p of N sampled i.i.d. random variables must approach as $N \to \infty$ (under some mild conditions expressed later in this section) one of three forms of asymptotic distributions [17], namely*

$$\Lambda_1(p) = \exp(-p^{-\alpha}), \quad p > 0; \quad (23)$$

$$\Lambda_2(p) = \exp(-(-p)^{\alpha}), \quad p \leq 0; \quad (24)$$

$$\Lambda_3(p) = \exp(-e^{-p}), \quad -\infty < p < \infty, \quad (25)$$

where $\alpha > 0$, and p is defined up to translation and scaling parameters.

We give here a short intuitive derivation of this result for self-containment sake, but refer the reader to [17] for a thorough discussion, and to [18] for a formal proof. We follow closely the discussion given in [17], which was originally introduced by Fisher and Tippett [19].

Consider taking the maximum value in a sample of $N = mn$; this amounts in taking maximum values in n different subset (each containing m samples), then taking the maximum of those n values. Obviously we need both distributions to converge to the same form $\Lambda(p)$ as $m \to \infty$. Since n remains finite, we use the fact that the cdf of the maximum value of n i.i.d. random variables with cdf $F_p(p_0) \equiv \Pr(p \leq p_0)$ is given by $(F_p(p_0))^n$, as each of the n values have to be smaller than p_0 for the maximum to be too. Therefore the asymptotic distribution $\Lambda(p)$ must be such that

$$\Lambda^n(a_n p + b_n) = \Lambda(p). \quad (26)$$

These asymptotic distributions play the same role for order statistics as the normal distribution for the sum of random variables, i.e. it is in most aspects analogous to the central limit theorem. Here $a_n > 0$ and b_n are suitably chosen normalization constants, just like the sum of n i.i.d. variables must be centered, and normalized by n in order to converge to a standard normal distribution. It is now easily checked that the three form of $\Lambda(p)$ presented above are the only solutions to equation (26) (see [17]).

When some minimal conditions on the underlying distribution are respected, the extremal value will converge to one of the three forms above; there exist known conditions on the underlying distribution (from which the maximum is taken) stating which one of the three forms is the asymptote, although in *most* cases it converges to the third form as conditions for convergence to the first two forms are more restrictive. These conditions were obtained by Gnedenko [18] and are given here in terms of the cdf $P(x)$ of the underlying distribution.

Proposition 2 *$P(x)$ belongs to the domain of attraction of Λ_1 if and only if*

$$\lim_{x\to\infty} \frac{1-P(x)}{1-P(kx)} = k^{\alpha}, \tag{27}$$

for every $k > 0$.

Proposition 3 *$P(x)$ belongs to the domain of attraction of Λ_2 if and only if there exists an x_0 such that*

$$P(x_0) = 1 \quad \text{and} \quad P(x_0 - \epsilon) < 1 \tag{28}$$

for every $\epsilon > 0$,
and if and only if

$$\lim_{x\to 0^-} \frac{1-P(kx+x_0)}{1-P(x+x_0)} = k^{\alpha}, \tag{29}$$

for every $k > 0$.

In essence, these conditions are equivalent to a power-law behavior of the tail of the cdf. We note that distributions which fall into the Λ_1 domain are unlimited on the right, while they are limited for Λ_2. Distributions belonging to the Λ_3 domain can occur in both cases; the formal conditions for Λ_3 are more intricate, but a sufficient condition in the case where $P(x)$ is unlimited on the right is given here.

Proposition 4 *$P(x)$ belongs to the domain of attraction of Λ_3 if it is less than 1 for every finite x, it is twice differentiable for at least every x greater than some x' and it respects:*

$$\lim_{x\to\infty} \frac{d}{dx}\left[\frac{1-P(x)}{P'(x)}\right] = 0. \tag{30}$$

It is trivially checked that any exponential form belongs to the domain of attraction of Λ_3.

Figure 1 shows the evolution of the distribution of the maximum of N gaussian variables with mean $\mu = 0$ and variance $\sigma^2 = 1$ as the number of variables N grows. The gaussian distribution is one example of an underlying distribution that has Λ_3 has its asymptote for the maximum value in a sample.

$\Lambda_1(p)$ and $\Lambda_2(p)$ are known as Weibull and Fréchet distributions, respectively; their stretched-exponential behavior is reminiscent of the pdf for velocity increments at small scales [5]. We begin by considering only the third form $\Lambda_3(p)$, and will come back to the two others later on.

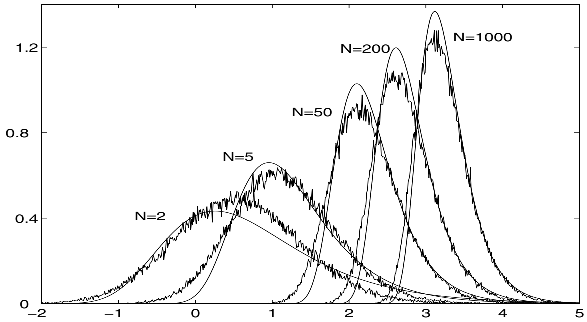

Fig. 1. Normalized histograms of the maximum value of N gaussian variables for $N = 2, 5, 50, 200, 1000$, along with the corresponding theoretical Λ_3 pdfs. For $N = 2$ the histogram is almost gaussian, as expected, while it tends to the asymmetrical Λ_3 form for higher values of N. Note the relatively slow convergence to the theoretical curves; this is characteristic of the gaussian law. Maxima of an exponential law converge faster. Each curve involves 100,000 realizations.

4.3 Form of τ(p)

In Section 3 we have related the moments $\langle \varepsilon_l^p \rangle$ to the cdf of p; from our interpretation of the entrainment process proposed by She and Lévêque we derived three possible forms for this cdf. This allows us to obtain specific forms for the dissipation scaling exponent $\tau(p)$.

We assume for now that $\Lambda_3(p)$ is the proper distribution for p. $\Lambda_3(p)$ is known as the Gumbel distribution; it is a special case of the larger class of Fisher-Tippett distribution, with additional parameters for translation and scaling of its argument. From equations (21) and (25) we can write

$$\langle \varepsilon_l^p \rangle = (\varepsilon_l^{(\infty)})^p \cdot \left[1 - \exp(-e^{-(ap-b)})\right], \tag{31}$$

where a and b may depend on l. We define C_∞ as

$$\varepsilon_l^{(\infty)} \sim l^{C_\infty}; \tag{32}$$

She and Lévêque assumed that $C_\infty = -2/3$. Hence for $\tau(p)$ we have

$$\tau(p) = C_\infty \cdot p + \lim_{l\to 0} \frac{\log\left[1 - \exp(-e^{-(ap-b)})\right]}{\log(l)}. \tag{33}$$

We are interested in the behavior of this limit. Note first that the double exponential term takes values between 0 and 1. Any limit value other than 1 for this double exponential forces the limit of the ratio to zero, since the $\log(l)$ term will dominate. This leaves only the linear term, thus in this case $\tau(p)$ would be a linear function of p. If the double exponential does indeed converge to 1, we can expand it in its Taylor series around $-e^{-(ap-b)} = 0$, resulting in

$$\tau(p) = C_\infty \cdot p + \lim_{l\to 0} \frac{-(ap-b)}{\log(l)}. \tag{34}$$

The limit in this case is non-trivial only if a or b are proportional to $\log(l)$, in which case the behavior for $\tau(p)$ is still limited to a linear form.

Similar arguments hold also for the two other max-stable asymptotes, Λ_1 and Λ_2.

At first, this comes as a bit of a disappointment since we expected to recover non-linear behavior for $\tau(p)$. Bluntly put, this is due to the presence of the $1 - \bar{P}(p)$ term rather than $\bar{P}(p)$, whereas the double exponential term could then yield exponential behavior for $\tau(p)$. We would need a double exponential behavior of $1 - \bar{P}(p)$ for this to be true. This is just the case with min-stable distribution. Indeed, from the fact that taking the minimum amounts to take the maximum after sign reversal, one readily sees that

$$\bar{P}_{\min}(p) = 1 - \bar{P}_{\max}(-p). \tag{35}$$

Before we discuss what meaning could take a min-stable distribution for p, we inspect some interesting results that are obtained with this distribution in the present context.

4.4 Obtaining the She-Lévêque universal scaling law from the (min-stable) Fisher-Tippett distribution

We now present a derivation of the She-Lévêque universal scaling law from a min-stable constraint on the distribution of structures.

Still working with Λ_3, the cdf of p is such that $1 - \bar{P}(p) = \exp(-e^{ap-b})$. Using equation (21), the intensity of p-order structures can then be written as

$$\varepsilon_l^{(p)} = \frac{\langle \varepsilon^{p+1} \rangle}{\langle \varepsilon^p \rangle} \tag{36}$$

$$= \frac{(\varepsilon_l^{(\infty)})^{p+1} \cdot \exp(-e^{ap-b})^{e^a}}{(\varepsilon_l^{(\infty)})^p \cdot \exp(-e^{ap-b})} \tag{37}$$

$$= \varepsilon_l^{(\infty)} \cdot \exp(-e^{ap-b})^{e^a - 1}. \tag{38}$$

We can now derive the following expression for $\varepsilon_l^{(p+1)}$ with respect to $\varepsilon_l^{(p)}$:

$$\varepsilon_l^{(p+1)} = \varepsilon_l^{(\infty)} \cdot \exp(-e^{a(p+1)-b})^{e^a - 1} \tag{39}$$

$$= (\varepsilon_l^{(\infty)})^{1-e^a} \cdot (\varepsilon_l^{(p)})^{e^a}. \tag{40}$$

But this is equivalent to equation (6) with $\beta = e^a$ and $A_p = 1$. Since $\beta = 2/3$ in the context of the She-Lévêque model [7], we get $a = \log(2/3)$, independent of l. Of course A_p is also independent of l. Hence we have obtained (6) from a simple order statistics argument, rather than having to postulate it[1].

It seems however that we have cheated by switching from a max-stable to a min-stable law. While we delay the full discussion to Section 5, we make the following comment to sooth the reader's discomfort. Even though the scaling law was postulated by She and Lévêque based upon the idea of an entrainment process, there is nothing that says, in its explicit formulation in equation (6), who "wins the battle" between the highly and the poorly organized structure. We shall make a stronger statement in Section 5. For now, we take a more general look at stable laws.

4.5 General study of the three asymptotic forms

We have just showed that we can obtain the She-Lévêque universal scaling law without having to postulate it, but rather from first principles based on order statistics and from some fundamental hypothesis forming the model. The main hypothesis was that there exists an entrainment process over the order p of the structures $\varepsilon_l^{(p)}$. We have replaced max-stable by min-stable statistics; we have also made the (unjustified) additional hypothesis that Λ_3 was the proper distribution of the extreme. In this section we will obtain more general results relying solely on the entrainment process, and on the independence of $\bar{\varepsilon} = \varepsilon_l^{(0)}$ over scale, for which there is strong experimental evidence.

In the She-Lévêque model, equation (6) was used to derive the behavior of exponent $\tau(p)$; the She-Lévêque model did not provide any explicit expression for the moments $\langle \varepsilon_l^p \rangle$, such as the one we just derived in equation (16), which we rewrite as

$$\langle \varepsilon_l^p \rangle = (\varepsilon_l^{(\infty)})^p \cdot (1 - \bar{P}(p)). \tag{42}$$

For Λ_3, we readily see that this leads to

$$\tau(p) = C_\infty \cdot p + \lim_{l \to 0} \frac{\log(\exp(-e^{-(ap-b)}))}{\log(l)} \tag{43}$$

$$= C_\infty \cdot p + \lim_{l \to 0} \frac{-e^{-(ap-b)}}{\log(l)}, \tag{44}$$

which, for proper scaling of a and b, gives the non-trivial form:

$$\tau(p) = C_\infty \cdot p - e^{-(a_0 p - b_0)}. \tag{45}$$

By proper scaling, we mean $a = a_0$ (independent of l) and $b = b_0 - \log(-\log(l))$, with $a_0 = b_0 = -\log(2/3)$ in the She-Lévêque model.

We now come back to the two other asymptotic forms, Λ_1 and Λ_2. The cdf for the second form is

$$\Lambda_2(ap - b) = 1 - \exp(-(ap - b)^\alpha), \tag{46}$$

where $\alpha > 0$ and $ap - b > 0$. $\Lambda_2(ap - b)$ is equal to 0 for $ap - b \leq 0$.

We can write the following for $\tau(p)$ in this case:

$$\tau(p) = C_\infty \cdot p + \lim_{l \to 0} \left(\frac{-(ap-b)^\alpha}{\log(l)} \right). \tag{47}$$

For $\alpha > 1$, even with proper scaling of a and b, this forms leads to a non-concave form for $\tau(p)$, which we discard. For $0 < \alpha < 1$, we have the following form $\tau(p)$: $\tau(p) = C_\infty . p + a_0 p^\alpha$. However, this form does not correspond to structures of codimension $C = 2$ for $p \to \infty$ due to the sublinear term $a_0 p^\alpha$.

Finally, for the Λ_1 form the normalizing rule is such that for positive values of p only a linear behavior of $\tau(p)$ is admissible. Hence, for both Λ_1 and Λ_2 forms, the only admissible behavior is K41.

Remark: Note that if we assume a standard averaging dynamics on the structures such that $P_p(p)$ is simply

[1] Finally we also have that $\varepsilon_l^{(0)}$ must not depend on l, such that

$$b = -\log(-\log(l^{2/3})) = -\log(2/3) - \log(-\log(l)). \tag{41}$$

Here l is non-dimensionalized by division by the integral scale l_0, such that l runs from 1 to 0.

gaussian, then with proper scaling of the mean μ and the variance σ^2 we obtain the log-normal model. However in this case we no longer have that $h(p) \to C_\infty$ as $p \to \infty$. Thisi s important since there is still a debate concerning whether or not there really exists experimental evidence that the scaling behavior of the dissipation differs from the log-normal model, see for instance [20].

5 Discussion and conclusion

We have presented a statistical interpretation of the She-Lévêque model allowing the derivation of the *universal scaling law* that was postulated in [7]. This derivation relies on constraints that must be respected by the system in order to behave dynamically as proposed in the She-Lévêque model. The constraints impose for the pdf of the structure order p to be one of three admissible asymptotic distributions for the maximum value in a neighborhood of a point. Out of those three valid distributions the Fisher-Tippett law is known to play a pre-eminent role because of its softer convergence conditions. The Fisher-Tippett law also happens to allow for the derivation of the conjectured scaling behavior of the energy dissipation as found in the work of She and Lévêque.

In the process, we have moved from max-stable to min-stable laws. We shall try to explain why it is necessary to do so here. As we mentioned earlier, the entrainment process does not tell us, per say, who wins the battle when it takes place. While the picture we have in mind is one of the most organized structures "dragging" along less organized fluid elements, it might very well be that the result of this process is to exhaust the structure of higher order. In other words, in the limit of large number of structures the entrainment process might be seen as a thermodynamical interaction, which should always favor entropy, i.e. structures indexed by lower p. This effect would be balanced by an intrinsic tendency to form structures, through the minimization of some generalized energy. This needs experimental validation.

All results were obtained by considering the entrainment process proposed originally by She and Lévêque, then describing its effect in terms of statistical constraints imposed on the intensity $\varepsilon_l^{(p)}$ of fluctuation structures. Obviously the entrainment process itself is understood as an additional hypothesis on the phenomenology of fully developed turbulence. Ideally this hypothesis could be understood as the manifestation of a hidden symmetry of the Navier-Stokes equation. As presented above the entrainment process amounts to invariance of a solution under replacement by the minimum in a neighborhood. This transformation can be loosely interpreted (for a scalar field) as a local translation in the direction opposite to the gradient, since the local minimum is bound to be found in this direction. Local maxima, minima and saddle points appear as fixed points of this transformation. Perhaps an additional symmetry of the Navier-Stokes dynamics could be formulated in a similar fashion.

The question also arises of how this could be verified experimentally, or at least on numerical data. We would like to propose a "regional" estimation of the moments $\langle \varepsilon_l^p \rangle$, and then use equation (16) to build the cdf of p. A proper definition of this estimation procedure is not trivial, and is the subject of ongoing work.

The author would like to thank Alain Arnéodo, Jean-Marc Lina and Frédéric Lesage for useful comments. The author also acknowledges insightful comments by the referees who revised a prior version of this paper.

References

1. A.N. Kolmogorov, C.R. Acad. Sci. USSR **30**, 301 (1941)
2. A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962)
3. F. Anselmet, Y. Gagne, E.J. Hopfinger, R.A. Antonia, J. Fluid Mech. **140**, 63 (1984)
4. A. Vincent, M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991)
5. U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995)
6. C. Meneveau, K.R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991)
7. Z.-S. She, E. Lévêque, Phys. Rev. Lett. **72**, 336 (1994)
8. R. Benzi, S. Ciliberto, C. Baudet, F. Massaioli, R. Tripiccione, S. Succi, Phys. Rev. E **48**, 29 (1993)
9. B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994)
10. E.A. Novikov, Phys. Rev. E **50**, 3303 (1995)
11. B. Castaing, B. Dubrulle, J. Phys. II France **5**, 895 (1995)
12. Z.-S. She, E.C. Waymire, Phys. Rev. Lett. **74**, 262 (1995)
13. Z.-S. She, E. Jackson, S.A. Orszag, Nature (London) **344**, 226 (1990)
14. Z.-S. She, E. Jackson, S.A. Orszag, Proc. R. Soc. London A **434**, 101 (1991)
15. Z.-S. She, Fluid Dyn. Res. **8**, 143 (1991)
16. J.-P. Bouchaud, A. Georges, Phys. Rep. **195 (4 & 5)**, 127 (1990)
17. H.A. David, *Order statistics*, 2nd edn., Wiley Series in Prob. and Math. Stat., Chapter 9 (Wiley, 1981)
18. B. Gnedenko, Ann. Math. Second Series **44 (3)**, 423 (1943)
19. R.A. Fisher, L.H.C. Tippett, Proc. Camb. Phil. Soc. **24**, 180 (1928)
20. J. Delour, J.F. Muzy, A. Arnéodo, Eur. Phys. J. B **23**, 243 (2001)